2. Differential Geometry of Curves

The differential geometry of curves and surfaces is fundamental in Computer Aided Geometric Design (CAGD). The curves and surfaces treated in differential geometry are defined by functions which can be differentiated a certain number of times. Books by Hilbert and Cohn-Vossen [165], Koenderink [205] provide intuitive introductions to the extensive mathematical literature on three-dimensional shape analysis. The books by Struik [412], Willmore [444], Kreyszig [206], Lipschutz [235], do Carmo [76] offer firm theoretical basis to the differential geometry aspects of three-dimensional shape description. A book by Gray [136] combines the traditional textbook style and a symbolic manipulation program MATHEMATICA. In a recent textbook, Gallier [122] provides a thorough introduction to differential geometry as well as a comprehensive treatment of affine and projective geometry and their applications to rational curves and surfaces in addition to basic topics of computational geometry (eg. convex hulls, Voronoi diagrams and Delaunay triangulations). We briefly review elementary differential geometry of curves in this chapter and surfaces in Chap. 3.

2.1 Arc length and tangent vector

Let us consider a segment of a parametric curve $\mathbf{r} = \mathbf{r}(t)$ between two points P $(\mathbf{r}(t))$ and Q $(\mathbf{r}(t+\Delta t))$ as shown in Fig. 2.1. Its length Δs can be approximated by a chord length $|\Delta\mathbf{r}| = |\mathbf{r}(t+\Delta t) - \mathbf{r}(t)|$, and by means of a Taylor expansion we have

$$\Delta s \simeq |\Delta\mathbf{r}| = |\mathbf{r}(t+\Delta t) - \mathbf{r}(t)| = \left|\frac{d\mathbf{r}}{dt}\Delta t + \frac{d^2\mathbf{r}}{dt^2}(\Delta t)^2\right| \simeq \left|\frac{d\mathbf{r}}{dt}\right|\Delta t\ , \quad (2.1)$$

to the first order approximation.

Thus as point Q approaches P or in other words $\Delta t \to 0$, the length Δs becomes the differential arc length of the curve:

$$ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = |\dot{\mathbf{r}}|dt = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}dt\ . \quad (2.2)$$

Here the dot $\dot{}$ denotes differentiation with respect to the parameter t. Therefore the arc length of a segment of the curve between points $\mathbf{r}(t_o)$ and $\mathbf{r}(t)$ can

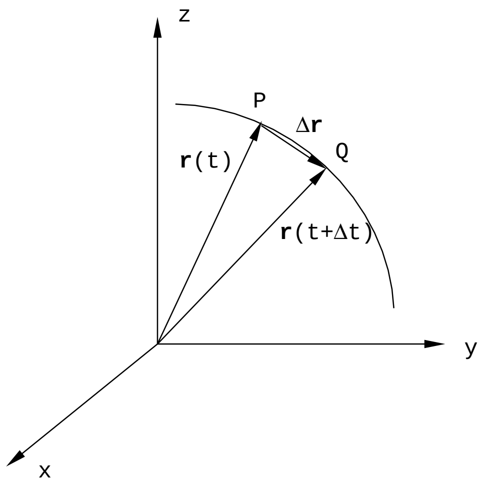

Fig. 2.1. A segment $\Delta\mathbf{r}$ connecting two point P and Q on a parametric curve $\mathbf{r}(t)$

be obtained as follows (provided the function $t \in [t_0, t] \rightarrow \mathbf{r}(t)$ is one-to-one almost everywhere):

$$s(t) = \int_{t_o}^{t} ds = \int_{t_o}^{t} \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} dt = \int_{t_o}^{t} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} dt \,. \tag{2.3}$$

The vector $\frac{d\mathbf{r}}{dt}$ is called the *tangent vector* at point P. This tangent vector has a simple geometrical interpretation. The vector $\mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ indicates the direction from $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$. If we divide the vector by Δt and take the limit as $\Delta t \rightarrow 0$, then the vector will converge to the finite magnitude vector $\dot{\mathbf{r}}(t)$, i.e. the tangent vector. The magnitude of the tangent vector is derived from (2.2) as

$$|\dot{\mathbf{r}}| = \frac{ds}{dt} \,, \tag{2.4}$$

hence the unit tangent vector becomes

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds} \equiv \mathbf{r}' \,. \tag{2.5}$$

Here the prime $'$ denotes differentiation with respect to the arc length. We will keep these notations, i.e. dot $\dot{}$ is for differentiation with respect to non-arc-length parameter t and prime $'$ with respect to arc length parameter s throughout the book. We list some useful formulae of the derivatives of arc length s with respect to parameter t and vice versa:

$$\dot{s} = \frac{ds}{dt} = |\dot{\mathbf{r}}| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \,, \tag{2.6}$$

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \;, \tag{2.7}$$

$$\dddot{s} = \frac{d\ddot{s}}{dt} = \frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}} + \ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{3}{2}}} \;, \tag{2.8}$$

$$t' = \frac{dt}{ds} = \frac{1}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \;, \tag{2.9}$$

$$t'' = \frac{dt'}{ds} = -\frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2} \;, \tag{2.10}$$

$$t''' = \frac{dt''}{ds} = -\frac{(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \dddot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) - 4(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{7}{2}}} \;. \tag{2.11}$$

Definition 2.1.1. *A regular (ordinary) point P on a parametric curve $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))^T$ is defined as a point where $|\dot{\mathbf{r}}(t)| \neq 0$. A point which is not a regular point is called a singular point.*

Definition 2.1.2. *A parametrization $\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t))^T$ of a curve defined in the interval I is called an allowable representation of class r [207], if it satisfies the following:*

1. *the mapping $\mathbf{r} : I \to \mathbf{R}^3$, $t \mapsto \mathbf{r}(t) = (x(t), y(t), z(t))^T$ is one-to-one,*
2. *the vector function $\mathbf{r} = \mathbf{r}(t)$ is of class $r \geq 1$ in the interval I,*
3. *$|\dot{\mathbf{r}}(t)| \neq 0$ for all $t \in I$.*

A parametric curve satisfying Definition 2.1.2 is also referred to as a *regular curve*. The magnitude of the tangent vector $\frac{ds}{dt}$ can be interpreted as a rate of change of the arc length s with respect to the parameter t and is called the *parametric speed*. If we assume the curve $\dot{\mathbf{r}}(t)$ to be regular, then by definition $|\dot{\mathbf{r}}(t)|$ is never zero and hence $\frac{ds}{dt}$ is always positive. When $\frac{ds}{dt} = 1$, the curve is said to be *arc length parametrized* or to have *unit speed*. If the parametric speed does not vary significantly, points of the curve obtained at parameter values $t_0, t_1, \cdots, t_N$ corresponding to a uniform increment $\Delta t = t_k - t_{k-1}$, will be nearly evenly distributed along the curve, as illustrated in Fig. 2.2. It is well known that every regular curve has an arc length parametrization [109], however, in practice it is very difficult to find it analytically, due to the fact that (2.3) is hard to integrate analytically. *Pythagorean hodograph* (*PH*) curves, introduced by Farouki and Sakkalis [108, 110], form a class of special planar polynomial curves whose parametric speed is a polynomial. Accordingly, its arc length is a polynomial function $s(t)$ of the parameter t. We provide a further review of Pythagorean hodograph curves and surfaces in Sect. 11.4.

Definition 2.1.3. *A point (x_0, y_0) of a planar irreducible implicit curve $f(x, y) = 0$ is said to be* singular *if $f(x_0, y_0) = f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.*

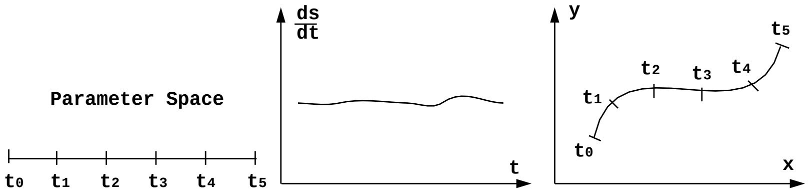

Fig. 2.2. When parametric speed does not vary significantly, points with uniformly spaced parameter values are nearly uniformly spaced along a parametric curve

The unit tangent vector for implicit curves can also be derived as follows. First we start with the planar curve $f(x, y) = 0$. The differential df of the implicit form $f = 0$ is zero, thus by letting $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ we have

$$df = f_x dx + f_y dy = 0 \; , \tag{2.12}$$

or assuming $f_y \neq 0$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \; . \tag{2.13}$$

Therefore the tangent vector on the implicit curve is given by $\pm(f_y, -f_x)^T$, and hence the unit tangent vector is

$$\mathbf{t} = \pm \frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}} \; . \tag{2.14}$$

The sign depends on the sense in which s increases.

As shown in Table 1.1, an implicit space curve is defined as the intersection of two implicit surfaces, $f(x, y, z) = 0$ and $g(x, y, z) = 0$. As we will see in Sect. 3.1, the normal vectors of these two implicit surfaces are ∇f and ∇g, respectively, where the symbol ∇ represents the gradient vector operator which is of the form $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$.

Since the tangent vector to the intersection curve is orthogonal to the normals of the two implicit surfaces, the unit tangent vector is given by

$$\mathbf{t} = \pm \frac{\nabla f \times \nabla g}{|\nabla f \times \nabla g|} \; , \tag{2.15}$$

provided that the denominator is nonzero ($\nabla f \neq \mathbf{0}$ and $\nabla g \neq \mathbf{0}$ or in other words the two surfaces are nonsingular and the surfaces are not tangent to each other at their common point under consideration). The unit tangent vector of the intersection of two implicit surfaces, when the two surfaces intersect tangentially is given in Sect. 6.4. Also here the sign depends on the sense in which s increases. A more detailed treatment of the tangent vector of implicit curves resulting from intersection of various types of surfaces can be found in Chap.6.

Example 2.1.1. The semi-cubical parabola, which is illustrated in Fig. 2.3, can be represented in parametric form as the curve $\mathbf{r}(t) = (t^2, t^3)^T$ [227]. The parametric speed is evaluated as $|\dot{\mathbf{r}}(t)| = \sqrt{t^2(4 + 9t^2)}$. It becomes zero when $t = 0$, hence it is singular at the origin and forms a cusp, which is illustrated in Fig. 2.3. The curve can be also represented implicitly $f(x, y) = x^3 - y^2 = 0$. We can also observe that $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$.

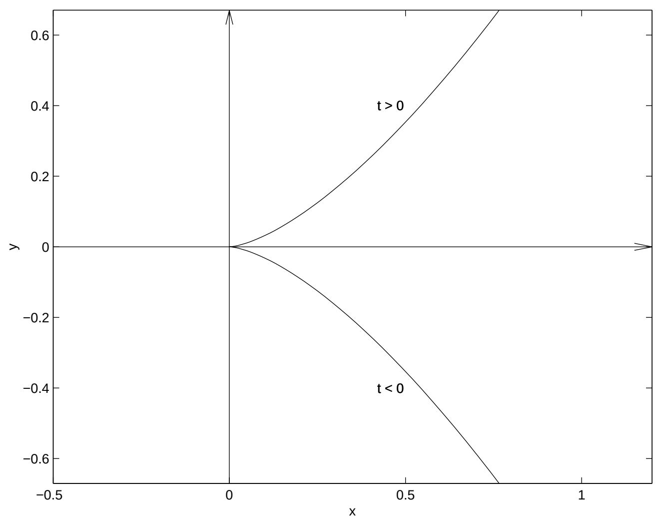

Fig. 2.3. A singular point occurs on a semi-cubical parabola in the form of a cusp

2.2 Principal normal and curvature

If $\mathbf{r}(s)$ is an arc length parametrized curve, then $\mathbf{r}'(s)$ is a unit vector (see (2.5)), and hence $\mathbf{r}' \cdot \mathbf{r}' = 1$. Differentiating this relation, we obtain

$$\mathbf{r}' \cdot \mathbf{r}'' = 0 \,, \tag{2.16}$$

which states that $\mathbf{r}''$ is orthogonal to the tangent vector, provided it is not a null vector. This fact can be also interpreted from the definition of the second derivative $\mathbf{r}''(s)$

$$\mathbf{r}''(s) = \lim_{\Delta s \to 0} \frac{\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)}{\Delta s} \,. \tag{2.17}$$

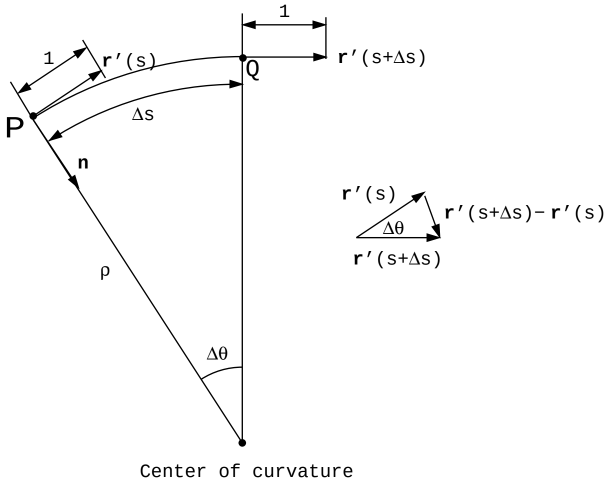

Fig. 2.4. Derivation of the normal vector of a curve (adapted from [455])

As shown in Fig. 2.4, the direction of $\mathbf{r}'(s+\Delta s)-\mathbf{r}'(s)$ becomes perpendicular to the tangent vector as $\Delta s \to 0$. The unit vector

$$\mathbf{n}=\frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|}=\frac{\mathbf{t}'(s)}{|\mathbf{t}'(s)|}\ , \tag{2.18}$$

which has the direction and sense of $\mathbf{t}'(s)$ is called the *unit principal normal vector* at s. The plane determined by the unit tangent and normal vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called the *osculating plane* at s. It is also well known that the plane through three consecutive points of the curve approaching a single point defines the osculating plane at that point [412].

When $\mathbf{r}'(s+\Delta s)$ is moved from Q to P, then $\mathbf{r}'(s)$, $\mathbf{r}'(s+\Delta s)$ and $\mathbf{r}'(s+\Delta s)-\mathbf{r}'(s)$ form an isosceles triangle (see Fig. 2.4), since $\mathbf{r}'(s+\Delta s)$ and $\mathbf{r}'(s)$ are unit tangent vectors. Thus we have $|\mathbf{r}'(s+\Delta s)-\mathbf{r}'(s)|=\Delta\theta\cdot 1=\Delta\theta=|\mathbf{r}''(s)\Delta s|$ as $\Delta s\to 0$ and hence

$$|\mathbf{r}''(s)|=\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s}=\lim_{\Delta s\to 0}\frac{\Delta\theta}{\varrho\Delta\theta}=\frac{1}{\varrho}\equiv\kappa\ . \tag{2.19}$$

κ is called the *curvature* , and its reciprocal ϱ is called the *radius of curvature* at s. It follows that

$$\mathbf{r}''=\mathbf{t}'=\kappa\mathbf{n}\ . \tag{2.20}$$

The vector $\mathbf{k}=\mathbf{r}''=\mathbf{t}'$ is called the *curvature vector*, and measures the rate of change of the tangent along the curve. By definition κ is nonnegative, thus the sense of the normal vector is the same as that of $\mathbf{r}''(s)$.

The curvature for arbitrary speed (non-arc-length parametrized) curve can be obtained as follows. First we evaluate $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ by the chain rule

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{t}v \ , \tag{2.21}$$

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\mathbf{t}v] = \frac{d\mathbf{t}}{ds}v^2 + \mathbf{t}\frac{dv}{dt} = \kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt} \ , \tag{2.22}$$

where $v = \frac{ds}{dt}$ is the parametric speed. Taking the cross product of $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ we obtain

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \kappa v^3 \mathbf{t} \times \mathbf{n} \ . \tag{2.23}$$

For the planar curve, we can give the curvature κ a sign by defining the normal vector such that $(\mathbf{t}, \mathbf{n}, \mathbf{e}_z)$ form a right-handed screw, where $\mathbf{e}_z = (0, 0, 1)^T$ as shown in Fig. 2.5. The point where the curvature changes sign is called an *inflection point* (see also Fig. 8.3).

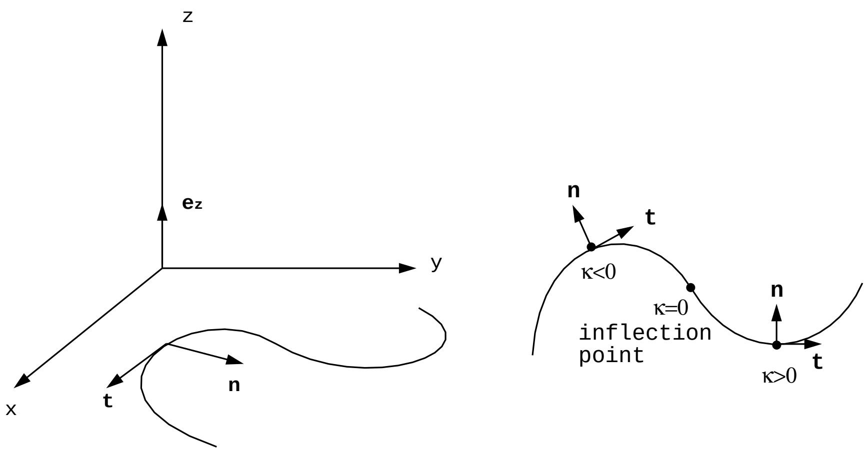

Fig. 2.5. Normal and tangent vectors along a 2D curve

According to this definition the unit normal vector of the plane curve is given by

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y}, \dot{x})^T}{\sqrt{\dot{x}^2 + \dot{y}^2}} \ , \tag{2.24}$$

and hence from (2.23) we have

$$\kappa = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \mathbf{e}_z}{v^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \ . \tag{2.25}$$

For a space curve, by taking the norm of (2.23) and using (2.4), we obtain

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \ . \tag{2.26}$$

The normal vector for the arbitrary speed curve can be obtained from $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, where $\mathbf{b}$ is the unit *binormal vector* which will be introduced in Sect. 2.3 (see (2.41)).

The unit principal normal vector and curvature for implicit curves can be obtained as follows. For the planar curve the normal vector can be deduced by combining (2.14) and (2.24) yielding

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(f_x, f_y)^T}{\sqrt{f_x^2 + f_y^2}} = \frac{\nabla f}{|\nabla f|} \ , \tag{2.27}$$

where only the + sign of $\mathbf{t}$ was used (although it is not necessary).

We will introduce a derivative operator with respect to arc length so that the derivation becomes simple. If we rewrite the plane implicit curve as $f(x(s), y(s)) = 0$ where s is arc length along the implicit curve, the total derivative with respect to the arc length becomes

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \ . \tag{2.28}$$

Now if we replace $\frac{dx}{ds}$ and $\frac{dy}{ds}$ by using (2.5) and (2.14) (+ sign), we obtain the derivative operator with respect to arc length

$$\frac{d}{ds} = \frac{1}{|\nabla f|}\left(f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y} \right) \ . \tag{2.29}$$

By applying the operator (2.29) to (2.14) (+ sign) and equating with $\kappa\mathbf{n}$ (using (2.20) and (2.27)), we obtain

$$\kappa = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}} \ . \tag{2.30}$$

For a 3-D implicit curve, we can deduce a derivative operator [444] similar to (2.29),

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z} \right) \ , \tag{2.31}$$

where $\boldsymbol{\alpha}$ is the tangent vector of the 3-D implicit curve (see (2.15)) given by

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = \nabla f \times \nabla g \ , \tag{2.32}$$

and

$$\alpha_1 = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z} , \tag{2.33}$$

$$\alpha_2 = \frac{\partial g}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} , \tag{2.34}$$

$$\alpha_3 = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} . \tag{2.35}$$

By applying the derivative operator (2.31) to $|\boldsymbol{\alpha}|\mathbf{t} = \boldsymbol{\alpha}$ we obtain

$$\frac{d|\boldsymbol{\alpha}|\mathbf{t}}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial \boldsymbol{\alpha}}{\partial z}\right) , \tag{2.36}$$

which gives

$$|\boldsymbol{\alpha}|^2\kappa\mathbf{n} + |\boldsymbol{\alpha}||\boldsymbol{\alpha}|'\mathbf{t} = \left(\alpha_1\frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial \boldsymbol{\alpha}}{\partial z}\right) . \tag{2.37}$$

Taking the cross product of $|\boldsymbol{\alpha}|\mathbf{t} = \boldsymbol{\alpha}$ and (2.37) yields

$$|\boldsymbol{\alpha}|^3\kappa\mathbf{b} = \boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial \boldsymbol{\alpha}}{\partial z}\right) . \tag{2.38}$$

Thus,

$$\kappa = \frac{\left|\boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial \boldsymbol{\alpha}}{\partial z}\right)\right|}{|\boldsymbol{\alpha}|^3} . \tag{2.39}$$

A different derivation of the curvature of a 3-D implicit curve is given in Sect. 6.3.2.

2.3 Binormal vector and torsion

In Sects. 2.1 and 2.2, we have introduced the tangent and normal vectors, which are orthogonal to each other and lie in the osculating plane. Let us define a unit binormal vector $\mathbf{b}$ such that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ form a right-handed screw, i.e.

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}, \qquad \mathbf{t} = \mathbf{n} \times \mathbf{b}, \qquad \mathbf{n} = \mathbf{b} \times \mathbf{t} , \tag{2.40}$$

which is shown in Fig. 2.6. The plane defined by normal and binormal vectors is called the *normal plane* and the plane defined by binormal and tangent vectors is called the *rectifying plane* (see Fig. 2.6). As mentioned before, the plane defined by tangent and normal vectors is called the *osculating plane*. The binormal vector for the arbitrary speed curve with nonzero curvature can be obtained by using (2.23) and the first equation of (2.40) as follows:

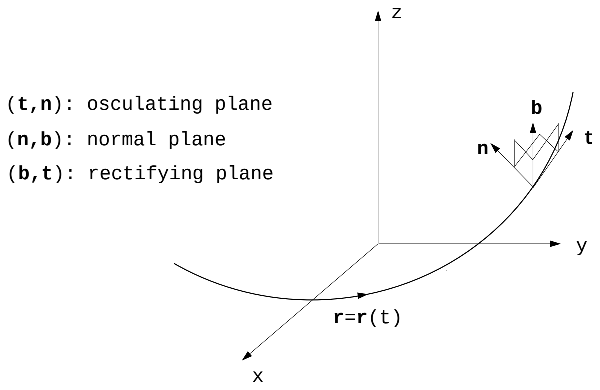

Fig. 2.6. The tangent, normal, and binormal vectors define an orthogonal coordinate system along a space curve

$$\mathbf{b} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|} \ . \tag{2.41}$$

The binormal vector is perpendicular to the osculating plane and its rate of change is expressed by the vector

$$\mathbf{b}' = \frac{d}{ds}(\mathbf{t} \times \mathbf{n}) = \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds} = \mathbf{t} \times \mathbf{n}' \ , \tag{2.42}$$

where we used the fact that $\frac{d\mathbf{t}}{ds} = \mathbf{r}'' = \kappa\mathbf{n}$.

Since $\mathbf{n}$ is a unit vector $\mathbf{n} \cdot \mathbf{n} = 1$, we have $\mathbf{n} \cdot \mathbf{n}' = 0$. Therefore $\mathbf{n}'$ is parallel to the rectifying plane $(\mathbf{b}, \mathbf{t})$, and hence $\mathbf{n}'$ can be expressed as a linear combination of $\mathbf{b}$ and $\mathbf{t}$:

$$\mathbf{n}' = \mu\mathbf{t} + \tau\mathbf{b} \ . \tag{2.43}$$

Thus, using (2.42) and (2.43), we obtain

$$\mathbf{b}' = \mathbf{t} \times (\mu\mathbf{t} + \tau\mathbf{b}) = \tau\mathbf{t} \times \mathbf{b} = -\tau\mathbf{b} \times \mathbf{t} = -\tau\mathbf{n} \ . \tag{2.44}$$

The coefficient τ is called the *torsion* and measures how much the curve deviates from the osculating plane. By taking the dot product with $-\mathbf{n}$, we obtain the torsion of the curve at a nonzero curvature point

$$\tau = -\mathbf{n} \cdot \mathbf{b}' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}''}{\kappa}\right)' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}'''}{\kappa}\right) = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}''} \ , \tag{2.45}$$

where (2.20) is used and $(\mathbf{r}'\mathbf{r}''\mathbf{r}''')$ is a *triple scalar product.* [1]

[1] A triple scalar product $(\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ is numerically equal to the volume of the parallelepiped having the edge vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and is given by

The torsion for an arbitrary speed curve is given by

$$\tau = \frac{(\dot{\mathbf{r}}\ddot{\mathbf{r}}\dddot{\mathbf{r}})}{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})} \; . \tag{2.48}$$

The evaluation of torsion when curvature vanishes is discussed in Sect. 6.2.

While the curvature is determined only in magnitude, except for plane curves, torsion is determined both in magnitude and sign. Torsion is positive when the rotation of the osculating plane is in the direction of a right-handed screw moving in the direction of $\mathbf{t}$ as s increases. If the torsion is zero at all points, the curve is planar.

The binormal vector of a 3-D implicit curve can be obtained from (2.38) as follows:

$$\mathbf{b} = \frac{\boldsymbol{\alpha}\times\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)}{|\boldsymbol{\alpha}\times\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)|} \; . \tag{2.49}$$

The torsion for a 3-D implicit curve can be derived by applying the derivative operator (2.31) to (2.38) [444], which gives

$$\begin{aligned}&\frac{d}{ds}(|\boldsymbol{\alpha}|^3\kappa\mathbf{b}) = \\ &\frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial}{\partial x}+\alpha_2\frac{\partial}{\partial y}+\alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha}\times\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right) \; ,\end{aligned} \tag{2.50}$$

and therefore

$$\begin{aligned}&|\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|^3\kappa)'\mathbf{b} - |\boldsymbol{\alpha}|^4\kappa\tau\mathbf{n} = \\ &\left(\alpha_1\frac{\partial}{\partial x}+\alpha_2\frac{\partial}{\partial y}+\alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha}\times\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right) \; .\end{aligned} \tag{2.51}$$

Taking the dot product with (2.37) we obtain

$$\begin{aligned}-|\boldsymbol{\alpha}|^6\kappa^2\tau = &\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right) \\ &\cdot\left(\alpha_1\frac{\partial}{\partial x}+\alpha_2\frac{\partial}{\partial y}+\alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha}\times\left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x}+\alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y}+\alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right) \; ,\end{aligned} \tag{2.52}$$

from which we calculate τ. An alternative approach for evaluating the torsion of 3-D implicit curves is presented in Sect. 6.3.3.

$$(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \; . \tag{2.46}$$

Also a cyclic permutation maintains the value of the triple scalar product:

$$(\mathbf{a}\,\mathbf{b}\,\mathbf{c}) = (\mathbf{b}\,\mathbf{c}\,\mathbf{a}) = (\mathbf{c}\,\mathbf{a}\,\mathbf{b}) \; . \tag{2.47}$$

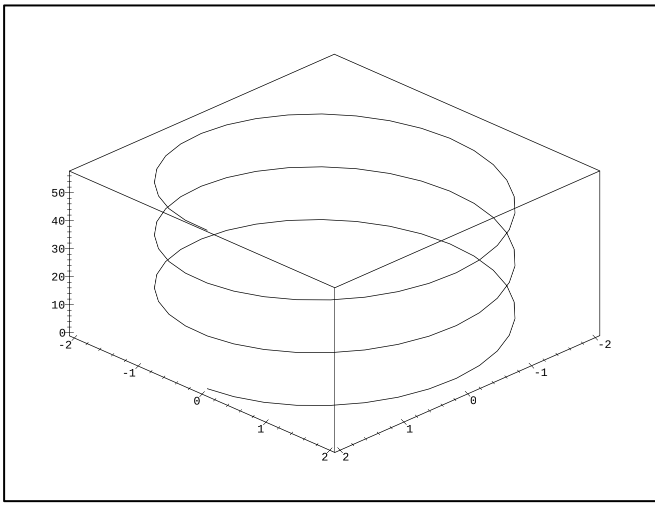

Fig. 2.7. Circular helix with $a = 2$, $b = 3$ for $0 \le t \le 6\pi$

Example 2.3.1. A circular helix in parametric representation is given by $\mathbf{r}(t) = (a\cos t, a\sin t, bt)^T$. Figure 2.7 shows a circular helix with $a = 2$, $b = 3$ for $0 \le t \le 6\pi$. The parametric speed is easily computed as $|\dot{\mathbf{r}}(t)| = \sqrt{a^2+b^2} \equiv c$, which is a constant. Therefore the curve is regular and its arc length is

$$s(t) = \int_0^t |\dot{\mathbf{r}}|dt = \int_0^t \sqrt{a^2+b^2}dt = ct\ .$$

We can easily reparametrize the curve with arc length by replacing t by $\frac{s}{c}$ yielding $\mathbf{r} = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c})^T$. The first three derivatives are evaluated as

$$\mathbf{r}'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)^T, \quad \mathbf{r}''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)^T,$$
$$\mathbf{r}'''(s) = \left(\frac{a}{c^3}\sin\frac{s}{c}, -\frac{a}{c^3}\cos\frac{s}{c}, 0\right)^T.$$

The curvature and torsion are evaluated as follows:

$$\kappa^2 = \mathbf{r}''\cdot\mathbf{r}'' = \frac{a^2}{c^4}\left(\cos^2\frac{s}{c} + \sin^2\frac{s}{c}\right) = \frac{a^2}{c^4} = \mathit{constant}\ ,$$
$$\tau = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}''\cdot\mathbf{r}''} = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\kappa^2} = \frac{c^4}{a^2}\begin{vmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\frac{a}{c^2}\cos\frac{s}{c} & -\frac{a}{c^2}\sin\frac{s}{c} & 0 \\ \frac{a}{c^3}\sin\frac{s}{c} & -\frac{a}{c^3}\cos\frac{s}{c} & 0 \end{vmatrix}$$

$$= \frac{c^4}{a^2}\frac{b}{c}\frac{a^2}{c^5}\left(\cos^2\frac{s}{c} + \sin^2\frac{s}{c}\right) = \frac{b}{c^2} = \textit{constant} \ .$$

Note that the circular helix has constant curvature and torsion and when $b > 0$, it is a right-handed helix while when $b < 0$, it is a left-handed helix.

2.4 Frenet-Serret formulae

From (2.20) and (2.44), we found that

$$\mathbf{t}' = \kappa\mathbf{n} \ , \tag{2.53}$$

$$\mathbf{b}' = -\tau\mathbf{n} \ . \tag{2.54}$$

From these equations we deduce

$$\mathbf{n}' = (\mathbf{b}\times\mathbf{t})' = \mathbf{b}'\times\mathbf{t} + \mathbf{b}\times\mathbf{t}' = -\tau\mathbf{n}\times\mathbf{t} + \mathbf{b}\times(\kappa\mathbf{n}) = -\kappa\mathbf{t} + \tau\mathbf{b} \ . \tag{2.55}$$

In matrix form we can express the differential equations as

$$\begin{pmatrix}\mathbf{t}'\\ \mathbf{n}'\\ \mathbf{b}'\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} \ . \tag{2.56}$$

Thus, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are completely determined by the curvature and torsion of the curve as a function of parameter s. The equations $\kappa = \kappa(s)$, $\tau = \tau(s)$ are called *intrinsic equations* of the curve. The formulae (2.56) are known as the Frenet-Serret formulae and describe the motion of a moving trihedron ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$) along the curve. From these $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ the shape of the curve can be determined apart for a translation and rotation. For arbitrary speed curve the Frenet-Serret formulae are given by

$$\begin{pmatrix}\dot{\mathbf{t}}\\ \dot{\mathbf{n}}\\ \dot{\mathbf{b}}\end{pmatrix} = \begin{pmatrix}0 & v\kappa & 0\\ -v\kappa & 0 & v\tau\\ 0 & -v\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} \ , \tag{2.57}$$

where $v = \frac{ds}{dt}$ is the parametric speed.

Example 2.4.1. As shown in Example 2.3.1 the intrinsic equations of circular helix are given by $\kappa(s) = \frac{a}{c^2}$, $\tau(s) = \frac{b}{c^2}$, where $c = \sqrt{a^2+b^2}$. In this example we derive the parametric equations of circular helix from these intrinsic equations. Substituting the intrinsic equations into the Frenet-Serret equations we obtain

$$\frac{d\mathbf{t}}{ds} = \frac{a}{c^2}\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\frac{a}{c^2}\mathbf{t} + \frac{b}{c^2}\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\frac{b}{c^2}\mathbf{n} \ .$$

We first differentiate the first equation twice and the second equation once with respect to s, which yield

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{a}{c^2}\frac{d\mathbf{n}}{ds}, \qquad \frac{d^3\mathbf{t}}{ds^3} = \frac{a}{c^2}\frac{d^2\mathbf{n}}{ds^2}, \qquad \frac{d^2\mathbf{n}}{ds^2} = -\frac{a}{c^2}\frac{d\mathbf{t}}{ds} - \frac{b^2}{c^4}\mathbf{n}\,,$$

where the third equation is used to replace $\frac{d\mathbf{b}}{ds}$. Eliminating $\mathbf{n}$, $\frac{d\mathbf{n}}{ds}$, $\frac{d^2\mathbf{n}}{ds^2}$ and recognizing that $\mathbf{t} = \frac{d\mathbf{r}}{ds}$, we obtain the fourth order differential equation

$$\frac{d^4\mathbf{r}}{ds^4} + \frac{1}{c^2}\frac{d^2\mathbf{r}}{ds^2} = 0\,.$$

The general solution to this differential equation is given by

$$\mathbf{r}(s) = \mathbf{C}_1 + \mathbf{C}_2 s + \mathbf{C}_3 \cos\frac{s}{c} + \mathbf{C}_4 \sin\frac{s}{c}\,,$$

where $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$ are the vector constants determined by the initial conditions. In this case we assume the following initial conditions

$$\mathbf{r}(0) = (a,0,0)^T, \qquad \mathbf{r}'(0) = \left(0, \frac{a}{c}, \frac{b}{c}\right)^T, \qquad \mathbf{r}''(0) = \left(-\frac{a}{c^2}, 0, 0\right)^T\,,$$
$$\mathbf{r}'''(0) = \left(0, -\frac{a}{c^3}, 0\right)^T,$$

which yield

$$\mathbf{C}_1 = (0,0,0)^T, \qquad \mathbf{C}_2 = \left(0,0,\frac{b}{c}\right)^T, \mathbf{C}_3 = (a,0,0)^T, \qquad \mathbf{C}_4 = (0,a,0)^T\,,$$

thus, we have $\mathbf{r}(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right)^T$.